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# MULTIPLE CRITICAL POINTS OF PERTURBED SYMMETRIC FUNCTIONALS

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ABSTRACT. Variational problems which are invariant under a group of symmetries often possess multiple solutions. This paper studies the effect of perturbations which are not small and which destroy the symmetry for two classes of such problems and shows how multiple solutions persist despite the perturbation.

During the past fifteen years there has been a considerable amount of research on the role of symmetry in obtaining multiple critical points of symmetric functionals both in an abstract setting and in applications to ordinary and partial differential equations. In particular, problems of the form,

$$\text{(0.1)} \qquad \begin{cases} Lu \equiv -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)u_{x_j}\right) + c(x)u = p(x,u), & x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{cases}$$

have been studied where $L$ is uniformly elliptic with e.g. $C^2$ coefficients, $\Omega \subset \mathbf{R}^n$ is a bounded domain with a smooth boundary, and $p$ is odd in $u$. Under appropriate hypotheses on $p(x, \xi)$, in particular more rapid growth than linear as $|\xi| \to \infty$, it has been shown that (0.1) possesses an unbounded sequence of solutions [**1**–**6**]. Similar existence statements have been obtained for periodic solutions of second order Hamiltonian systems of ordinary differential equations,

$$\text{(0.2)} \qquad \ddot{q} + V'(q) = 0,$$

where $q = (q_1, \ldots, q_n)$ and $V \in C^1(\mathbf{R}^n, \mathbf{R})$. Indeed, it has been shown that if $V$ grows at an appropriate superquadratic rate, then for any $T > 0$, (0.2) possesses an unbounded sequence of $T$ periodic solutions [**7**, **8**].

Problems (0.1) and (0.2) each possess a natural symmetry. Namely (0.1) is the Euler equation obtained from

$$\text{(0.3)} \qquad \int_\Omega \left[\frac{1}{2}\left(\sum_{i,j=1}^{n} a_{ij}(x)u_{x_i}u_{x_j} + c(x)u^2\right) - P(x,u)\right] dx$$

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where $P$, the primative of $p$, is even in $u$ and therefore the functional is invariant under the $\mathbf{Z}_2$ symmetry $u \to -u$. Likewise, taking e.g. $T = 2\pi$, we see (0.2) is the Euler equation of the functional

$$\int_0^{2\pi}\left[\frac{1}{2}|\dot{q}|^2 - V(q)\right] dt \tag{0.4}$$

defined on the class of $2\pi$ periodic functions. If $q(t) \to q(t+\theta)$ for any $\theta \in \mathbf{R}$, the functional is unchanged. Thus (0.4) has a natural $\mathbf{R}$ mod$[0, 2\pi]$ or $S^1$ symmetry.

An open question for problems like (0.1) and (0.2) has been the effect of destroying the above symmetries by perturbing the equation, e.g. by adding an inhomogeneous term $f(x)$ to the right-based side of (0.1) or a $2\pi$ periodic $n$-vector $\varphi(t)$ to the right-hand side of (0.2). There has been some progress in this direction during the past few months due to Bahri and Berestycki [**9**], Struwe [**10**], Dong and Li [**11**], and Bahri [**12**]. In [**9** and **10**], the authors independently show that

$$\begin{cases} Lu = p(x,u) + f(x), & x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{cases} \tag{0.5}$$

possesses an unbounded sequence of weak solutions provided that $f \in L^2(\Omega)$ and $p$ satisfies more stringent conditions than are required for the existence of solutions of (0.1). In this paper we shall show how some of the ideas from [**5**] in conjunction with those of [**9** and **10**] can be used to get somewhat better existence results for (0.5). Moreover closely related arguments allow us to treat perturbations of (0.2) of the form

$$q + V'(q) = \varphi(t). \tag{0.6}$$

Bahri and Berestycki [**13**] have also recently announced related results for (0.6).

In §1, (0.5) will be treated and (0.6) will be handled in §2. An appendix contains some topological results required for the study of (0.6). We are indebted to Ed Fadell and Sufian Husseini for several helpful conversations concerning these topological matters.

**1. The semilinear elliptic case.** We begin by studying

$$\begin{cases} Lu = p(x,u) + f(x), & x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $L$ and $\Omega$ are in the introduction. We assume $p$ satisfies

$(\mathrm{p}_1)$ $p \in C(\bar{\Omega} \times \mathbf{R}, \mathbf{R})$,

$(\mathrm{p}_2)$ there are constants $a_1, a_2 > 0$ such that $|p(x,\xi)| \leq a_1 + a_2|\xi|^s$ where $1 < s < (n+2)/(n-2)$ if $n > 2$ and $s$ is unrestricted if $n = 1, 2$,

$(\mathrm{p}_3)$ there are constants $\mu > 2$ and $\bar{\xi} > 0$ such that $0 < \mu P(x,\xi) \equiv \mu\int_0^\xi p(x,t)\,dt \leq \xi p(x,\xi)$ for $|\xi| \geq \bar{\xi}$,

$(\mathrm{p}_4)$ $p(x, -\xi) = -p(x,\xi)$.

Under hypotheses $(\mathrm{p}_1)$–$(\mathrm{p}_4)$, if $f \equiv 0$, it is known that (1.1) possesses an unbounded sequence of weak solutions which can be obtained as critical points of a corresponding functional by means of minimax methods. We shall show that the same is true for (1.1) for arbitrary $f \in L^2$ provided that $s$ satisfies the more stringent condition

$$\frac{(n+2)-(n-2)s}{n(s-1)} > \frac{\mu}{\mu-1}. \tag{1.2}$$

For $\mu = s+1$, (1.2) reduces to the assumption made in [**9** and **10**]. In [**9** and **10**] somewhat stronger versions of $(p_1)$ and $(p_3)$ also are needed as well as the requirement that $p$ behave like a positive function of $x$ times a pure power of $\xi$ at infinity.

For future reference we note that $(p_3)$ implies there are constants $a_3$, $a_4$, $a_5$ such that

$$\frac{1}{\mu}(\xi p(x,\xi) + a_3) \geqslant P(x,\xi) + a_4 \geqslant a_5|\xi|^\mu \tag{1.3}$$

for all $\xi \in \mathbf{R}$. Corresponding to (1.1) we have the functional

$$I(u) = \int_\Omega \left[\frac{1}{2}\left(\sum_{i,j=1}^n a_{ij}(x)u_{x_i}u_{x_j} + c(x)u^2\right) - P(x,u) - f(x)u\right] dx. \tag{1.4}$$

Letting $E = W_0^{1,2}(\Omega)$ where the norm in $E$ is

$$\|u\| = \left(\int_\Omega \sum_{i,j=1}^n a_{ij}(x)u_{x_i}u_{x_j}\,dx\right)^{1/2}$$

hypotheses $(p_1)$–$(p_2)$ and standard results imply $I \in C^1(E,\mathbf{R})$ (provided $f \in L^2(\Omega)$).

The main result in this section is

THEOREM 1.5. *Suppose $p$ satisfies* $(p_1) - (p_4)$, $f \in L^2(\Omega)$, *and $s$ satisfies* (1.2). *Then $I$ has an unbounded sequence of critical values.*

The corresponding critical points form an unbounded sequence of weak solutions of (1.1). Under additional regularity assumptions on $p$ and $f$ (e.g. $p$ and $f$ Hölder continuous in their arguments), standard regularity results imply these weak solutions are classical solutions of (1.1). After proving Theorem 1.5, a mild generalization of recent work of Dong and Li [**11**] in which $f$ is allowed to depend also on $u$ will be mentioned.

In the course of the proof of Theorem 1.5, we will obtain a minimax characterization of critical values of $I$ albeit not a completely satisfactory one. In [**9** and **10**], the fact that any solution of (1.1) lies on the set of $w$ in $E$ such that

$$\int_\Omega \left(\sum_{i,j=1}^n a_{ij}(x)w_{x_i}w_{x_j} + c(x)w^2\right) dx = \int_\Omega w(p(x,w) + f(x))\,dx \tag{1.6}$$

is exploited. On this set $I$ becomes

$$I(u) = \int_\Omega \left(\frac{1}{2}up(x,u) - P(x,u) - \frac{1}{2}f(x)u\right) dx \tag{1.7}$$

which is bounded from below. (See also [**1** and **2**].) We will work with an indefinite functional. However for technical reasons $I$ will be replaced by a modified functional $J$. By way of motivation for the modification, the following lemma provides some a prior bounds for a critical point of $I$ in terms of the corresponding critical value. In what follows $a_i$, $\alpha_j$ repeatedly denote positive constants.

LEMMA 1.8. *Suppose $u$ is a critical point of $I$. Then there is a constant $a_6$ depending on $\| f \|_{L^2}$ such that*

$$\int_\Omega (P(x,u) + a_4)\,dx \leq \frac{1}{\mu}\int_\Omega (up(x,u) + a_3)\,dx \leq a_6\big((I(u))^2 + 1\big)^{1/2}. \tag{1.9}$$

PROOF. Let $I'(u)$ denote the Fréchet derivative of $u$. At a critical point of $I$ we have

$$\begin{aligned} I(u) &= I(u) - \frac{1}{2}I'(u)u \\ &\geq \left(\frac{1}{2} - \frac{1}{\mu}\right)\int_\Omega (up(x,u) + a_3)\,dx - \frac{1}{2}\| f \|_{L^2}\| u \|_{L^2} - a_7 \end{aligned} \tag{1.10}$$

via (1.3). Using the fact that $\mu > 2$ and the Hölder and Young inequalities we see for any $\varepsilon > 0$,

$$I(u) \geq a_8\int_\Omega (up(x,u) + a_3)\,dx - a_9 - \varepsilon\| u \|^{\mu}_{L^\mu} - \beta(\varepsilon)\| f \|^{\nu}_{L^2} \tag{1.11}$$

where $\nu^{-1} + \mu^{-1} = 1$ and $\beta(\varepsilon) \to \infty$ as $\varepsilon \to 0$. Choosing $\varepsilon$ so that $2\varepsilon = \mu a_5 a_8$, (1.3), (1.11) and the Schwarz inequality yield (1.9).

REMARK 1.12. The inequality (1.11) combined with $I'(u)u = 0$ leads to a bound for $\| u \|$ in terms of $I(u)$. However such an estimate will not be needed later.

To define the modified functional $J$, let $\chi \in C^\infty(\mathbf{R}, \mathbf{R})$ such that $\chi(t) = 1$ for $t \leq 1$, $\chi(t) \equiv 0$ for $t > 2$ and $-2 < \chi' < 0$ for $t \in (1,2)$. For $u \in E$, set

$$\mathcal{G}(u) = 2a_6(I^2(u) + 1)^{1/2} \quad \text{and} \quad \psi(u) = \chi\left(\mathcal{G}(u)^{-1}\int_\Omega (P(x,u) + a_4)\,dx\right).$$

Let supp $\psi$ denote the support of $\psi$.

LEMMA 1.13. *If $u \in$ supp $\psi$, then*

$$\left|\int_\Omega fu\,dx\right| \leq \alpha_1(|I(u)|^{1/\mu} + 1) \tag{1.14}$$

*where $\alpha_1$ depends on $\| f \|_{L^2}$.*

PROOF. By the Schwarz and Hölder inequalities and (1.3), for all $u \in E$

$$\left|\int_\Omega fu\,dx\right| \leq \|f\|_{L^2}\|u\|_{L^2} \leq \alpha_2\|u\|_{L^\mu} \leq \alpha_3\left(\int_\Omega (P(x,u) + a_4)\,dx\right)^{1/\mu}. \tag{1.15}$$

If further $u \in$ supp $\psi$,

$$\int_\Omega (P(x,u) + a_4)\,dx \leq 4a_6(I^2(u) + 1)^{1/2} \leq \alpha_3(|I(u)| + 1) \tag{1.16}$$

so (1.14) follows from (1.15)–(1.16).

Now set

$$(1.17)\quad J(u)=\int_\Omega\left[\frac{1}{2}\left(\sum_{i,j=1}^{n}a_{ij}(x)u_{x_i}u_{x_j}+c(x)u^2\right)-P(x,u)-\psi(u)f(x)u\right]dx.$$

The main reason for introducing $J$ is the following estimate which holds for $J$ but not for $I$.

LEMMA 1.18. *There is a constant $\beta_1$, depending on $\| f \|_{L^2}$ such that for $u \in E$:*

$$(1.19)\qquad |J(u)-J(-u)| \leqslant \beta_1\left(|J(u)|^{1/\mu}+1\right).$$

PROOF. By (1.17) and ($p_4$),

$$(1.20)\qquad |J(u)-J(-u)| = (\psi(u)+\psi(-u))\left|\int_\Omega fu\,dx\right|.$$

Thus by Lemma 1.13,

$$(1.21)\qquad \psi(-u)\left|\int_\Omega fu\,dx\right| \leqslant \alpha_1\psi(-u)\left(|I(u)|^{1/\mu}+1\right).$$

Since by (1.4) and (1.17),

$$(1.22)\qquad |I(u)| \leqslant |J(u)| + 2\left|\int_\Omega fu\,dx\right|,$$

(1.21) implies that

$$(1.23)\qquad \psi(-u)\left|\int_\Omega fu\,dx\right| \leqslant \alpha_2\psi(-u)\left(|J(u)|^{1/\mu}+\left|\int_\Omega fu\,dx\right|^{1/\mu}+1\right).$$

Thus by Young's inequality as in (1.11), the $fu$ term on the right-hand side of (1.23) can be absorbed by the left-hand side. A corresponding estimate for the $\psi(u)$ term in (1.20) then yields (1.19).

REMARK 1.24. Although $I(u)$ does not satisfy (1.19) for all $u \in E$, it does for all solutions of (1.1). However we are unable to exploit this fact directly.

We shall show that large critical values of $J$ are critical values of $I$. First another technical lemma is needed.

LEMMA 1.25. *There are constants $M_0$, $\alpha_0 > 0$, and depending on $\| f \|_{L^2}$ such that whenever $M \geqslant M_0$, $J(u) \geqslant M$ and $u \in \operatorname{supp}\psi$, then $I(u) \geqslant \alpha_0 M$.*

PROOF. Since by (1.4) and (1.17)

$$(1.26)\qquad I(u) \geqslant J(u) - \left|\int_\Omega fu\,dx\right|,$$

if $u \in \operatorname{supp}\psi$, by (1.26) and (1.14)

$$(1.27)\qquad I(u)+\alpha_1|I(u)|^{1/\mu} \geqslant J(u)-\alpha_1 \geqslant M/2$$

for $M_0$ large enough. If $I(u) \leqslant 0$,

$$(1.28)\qquad \frac{\alpha_1^\nu}{\nu}+\frac{1}{\mu}|I(u)| \geqslant \alpha_1|I(u)|^{1/\mu} \geqslant \frac{M}{2}+|I(u)|$$

which is impossible if $M_0 > 2\alpha_1^\nu \nu^{-1}$ which we can assume to be the case. Therefore $I(u) > 0$ and

$$I(u) > M/4 \quad \text{or} \quad I(u) \geq \left(M(4\alpha_1)^{-1}\right)^{\mu}$$

which implies the lemma since $\mu > 2$.

Now we can prove

LEMMA 1.29. *There is a constant $M_1 > 0$ such that $J(u) \geq M_1$ and $J'(u) = 0$ implies that $J(u) = I(u)$ and $I'(u) = 0$.*

PROOF. It suffices to show that $\psi(u) = 1$ and $\psi'(u) = 0$. By the definition of $\psi$, this will be the case if

$$\mathcal{G}(u)^{-1}\int_\Omega (P(x,u) + a_4)\,dx \leq 1 \tag{1.30}$$

which we will verify. Note that

$$J'(u)u = \int_\Omega \left[ \sum_{i,j=1}^{n} a_{ij}(x)u_{x_i}u_{x_j} + c(x)u^2 - up(x,u) \right. \\ \left. - (\psi(u) + \psi'(u)u)fu \right] dx \tag{1.31}$$

where

$$\psi'(u)u = \chi'\left(\mathcal{G}(u)^{-1}\int_\Omega (P(x,u) + a_4)\,dx\right)$$
$$\times \mathcal{G}(u)^{-2}\left[\mathcal{G}(u)\int_\Omega up(x,u)\,dx - (2a_6)^2\left(\int_\Omega (P(x,u) + a_4)\,dx\right)\mathcal{G}(u)^{-1}I(u)I'(u)u\right].$$

Regrouping terms shows that

$$J'(u)u = (1 + T_1(u))\int_\Omega \left(\sum_{i,j=1}^{n} a_{ij}(x)u_{x_i}u_{x_j} + c(x)u^2\right) dx$$
$$- (1 + T_2(u))\int_\Omega up(x,u)dx - (\psi(u) + T_1(u))\int_\Omega fu\,dx$$

where

$$\begin{cases} T_1(u) = \chi'(\cdots)(2a_6)^2\mathcal{G}(u)^{-3}I(u)\int_\Omega (P(x,u) + a_4)\,dx\int_\Omega fu\,dx, \\ T_2(u) = \chi'(\cdots)\left[\mathcal{G}(u)^{-1}\int_\Omega fu\,dx\right] + T_1(u). \end{cases}$$

Form

$$J(u) - \frac{1}{2(1 + T_1(u))}J'(u)u.$$

If $T_1(u) = T_2(u) = 0$ and $\psi(u) = 1$ we are precisely in the situation of (1.10) so (1.30) reduces to (1.9). If $T_1(u)$ and $T_2(u)$ are sufficiently small, the estimates of

Lemma 1.8 carry over to this case at the expense of the factor $a_6$ in (1.9) being replaced by $2a_6$. But that gives (1.30). Hence the lemma follows once we show $T_1(u)$, $T_2(u) \to 0$ as $M_1 \to \infty$.

Simple estimates show

$$|T_1(u)| \leqslant |\chi'(\cdots)| 4a_6 \mathcal{I}(u)^{-1} \left| \int_\Omega fu\, dx \right|. \tag{1.32}$$

If $u \notin \operatorname{supp} \psi$, $T_1(u) = 0$. Otherwise, by Lemmas 1.13 and 1.25,

$$|T_1(u)| \leqslant \alpha_2 \mathcal{I}(u)^{(1/\mu)-1} \leqslant \alpha_3 (M_1 + 1)^{(1/\mu)-1}, \tag{1.33}$$

which goes to 0 as $M_1 \to \infty$. The form of $T_2$ shows $T_2(u)$ also $\to 0$ as $M_1 \to \infty$.

By Lemma 1.29 to prove Theorem 1.5, it suffices to show $J$ has an unbounded sequence of critical values. To begin that program, another technical result is required. Let $A_c = \{u \in E \mid J(u) \leqslant c\}$. We say $J$ satisfies the Palais-Smale condition (PS) if whenever a sequence $(u_m)$ satisfies $J(u_m)$ is uniformly bounded and $J'(u_m) \to 0$, then $(u_m)$ is precompact.

LEMMA 1.34. *$J \in C^1(E, \mathbf{R})$ and there is a constant $M_2 > 0$ such that $J$ satisfies* (PS) *on $\hat{A}_{M_2} \equiv \{u \in E \mid J(u) \geqslant M_2\}$.*

PROOF. Since $p$ satisfies ($p_1$), ($p_2$), $I \in C^1(E, \mathbf{R})$. (See e.g. [**14**].) Since $\chi \in C^\infty$, ($p_1$), ($p_2$) further imply $\psi$ and therefore $J \in C^1(E, \mathbf{R})$. To verify (PS), we argue somewhat as in Lemma 1.29. Suppose $(u_m) \subset E$ with $M_2 \leqslant J(u_m) \leqslant K$ and $J'(u_m) \to 0$. Then for all large $m$,

$$\begin{aligned} \rho \|u_m\| + K &\geqslant J(u_m) - \rho J'(u_m) u_m \\ &= \left(\frac{1}{2} - \rho(1 + T_1(u_m))\right) \int_\Omega \left( \sum_{i,j=1}^n a_{ij}(x) u_{mx_i} u_{mx_j} + c u_m^2 \right) dx \\ &\quad + \rho(1 + T_2(u_m)) \int_\Omega u_m p(x, u_m)\, dx - \int_\Omega P(x, u_m)\, dx \\ &\quad + \left[\rho(\psi(u_m) + T_1(u_m)) - \psi(u_m)\right] \int_\Omega f u_m\, dx \end{aligned} \tag{1.35}$$

where $\rho$ is free for the moment. Thus by ($p_3$),

$$\begin{aligned} \rho \|u_m\| + K &\geqslant \left(\tfrac{1}{2} - \rho(1 + T_1(u_m))\right) \|u_m\|^2 \\ &\quad + \left(\rho(1 + T_2(u_m))\mu - 1\right) \int_\Omega \left(P(x, u_m) + a_4\right) dx \\ &\quad - \alpha_2 - \left(\rho(1 + T_1(u_m)) + 1\right) \|f\|_{L^2} \|u_m\|_{L^2} \\ &\quad - \alpha_3 \|u_m\|_{L^2}^2. \end{aligned} \tag{1.36}$$

For $M_2$ sufficiently large and therefore $T_1$, $T_2$ sufficiently small, by ($p_3$) we can choose $\rho \in (\mu^{-1}, 2^{-1})$ and $\varepsilon > 0$ such that

$$\frac{1}{2(1 + T_1(u_m))} > \rho + \varepsilon > \rho - \varepsilon > \frac{1}{\mu(1 + T_2(u_m))} \tag{1.37}$$

uniformly in $m$. Hence (1.36) and (1.3) show

$$\rho\|u_m\| + K \geq \varepsilon\|u_m\|^2 + \frac{\varepsilon}{2}\mu a_5\|u_m\|_{L^\mu}^\mu - \alpha_2 - \alpha_4\|u_m\| - \alpha_3\|u_m\|_{L^2}^2. \tag{1.38}$$

Using the Hölder and Young inequalities again as in (1.11) we get

$$\rho\|u_m\| + K \geq \varepsilon\|u_m\|^2 + \alpha_5\|u_m\|_{L^\mu}^\mu - \alpha_4\|u_m\| - \alpha_6 \tag{1.39}$$

which implies $\{u_m\}$ is bounded in $E$.

Since

$$J'(u_m) = (1 + T_1(u_m))u_m - \mathcal{P}(u_m) \tag{1.40}$$

where $\mathcal{P}$ is compact—see e.g. [5]—for $M_2$ large enough $|T_1(u_m)| \leq \frac{1}{2}$ and therefore $(u_m)$ bounded and $J'(u_m) \to 0$ implies $(1 + T_1(u_m))^{-1}\mathcal{P}(u_m)$ converges along a subsequence. Hence (1.40) shows $(u_m)$ does also and (PS) is verified.

Now we can show $J$ has an unbounded sequence of critical values. Let $0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$ denote the eigenvalues of

$$\begin{cases} (L - c)v = \lambda v, & x \in \Omega, \\ v = 0, & x \in \partial\Omega, \end{cases} \tag{1.41}$$

and $v_1, v_2, \cdots$ denote corresponding eigenfunctions normalized such that $\|v_k\| = 1$. Let $E_k \equiv \operatorname{span}\{v_1, \ldots, v_k\}$ and $E_k^\perp$ its orthogonal complement. By (1.3) there is an $R_k > 0$ such that $J|_{E_k} \leq 0$ if $\|u\| \geq R_k$. Let $B_R$ denote the closed ball of radius $R$ in $E$, $D_k \equiv B_{R_k} \cap E_k$, and

$$\Gamma_k = \{h \in C(D_k, E) \mid h \text{ is odd and } h(u) = u \text{ if } \|u\| = R_k\}. \tag{1.42}$$

Define

$$b_k = \inf_{h \in \Gamma_k} \max_{u \in D_k} J(h(u)), \qquad k \in \mathbf{N}. \tag{1.43}$$

If $f \equiv 0$ and $J$ is even, it was essentially shown in [5] that the numbers $b_k$ are critical values of $J$. If $f \not\equiv 0$, that need not be the case. However we will use these numbers as the basis for a comparison argument to prove Theorem 1.5. First it will be shown that $b_k \to \infty$ as $k \to \infty$.

LEMMA 1.44. *For all $k \in \mathbf{N}$, $\rho < R_k$, and $h \in \Gamma_k$,*

$$h(D_k) \cap \partial B_\rho \cap E_{k-1}^\perp \neq \varnothing. \tag{1.45}$$

PROOF. Let $h \in \Gamma_k$. Consider $h^{-1}(B_\rho)$. Since $h$ is continuous, this is a neighborhood of 0 in $E_k$. Let $\mathcal{O}$ denote the component of $h^{-1}(B_\rho)$ which contains 0. Then $\mathcal{O} \subset D_k$, $\mathcal{O}$ is symmetric with respect to the origin, i.e. $u \in \mathcal{O}$ implies $-u \in \mathcal{O}$, and $\|h(u)\| = \rho$ on $\partial\mathcal{O}$. Let $P_{k-1}$, $P_{k-1}^\perp$ denote respectively the orthogonal projections of $E$ onto $E_{k-1}$, $E_{k-1}^\perp$ so $h(u) = P_{k-1}h(u) + P_{k-1}^\perp h(u)$ for $u \in E$. Since $P_{k-1}h \in C(\partial\mathcal{O}, E_{k-1})$ and is an odd function, by one of the versions of the Borsuk-Ulam Theorem [16], $P_{k-1}h$ has a zero $\hat{u}$ on $\partial\mathcal{O}$. Hence $h(\hat{u}) = P_{k-1}^\perp h(\hat{u}) \in \partial B_\rho \cap E_{k-1}^\perp$ and the proof is complete.

LEMMA 1.46. *There are constants $\beta_2 > 0$ and $k_0 \in \mathbf{N}$ depending on $\|f\|_{L^2}$ such that for all $k \geq k_0$,*

$$b_k \geq \beta_2 k^{((n+2)-(n-2)s)/n(s-1)}. \tag{1.47}$$

PROOF. Let $h \in \Gamma_k$ and $\rho < R_k$. By Lemma 1.44, there is a $w \in h(D_k) \cap \partial B_\rho \cap E_{k-1}^\perp$. Therefore

$$\max_{u \in D_k} J(h(u)) \geq J(w) \geq \inf_{u \in \partial B_\rho \cap E_{k-1}^\perp} J(u). \tag{1.48}$$

Let $u \in \partial B_\rho \cap E_{k-1}^\perp$. Then by ($p_2$) and some simple estimates,

$$J(u) \geq \tfrac{1}{2}\rho^2 - \alpha_2 \|u\|_{L^2}^2 - \alpha_3 \|u\|_{L^{s+1}}^{s+1} - \alpha_4 - \|f\|_{L^2} \|u\|_{L^2}. \tag{1.49}$$

By the Gagliardo-Nirenberg inequality [**17**],

$$\|u\|_{L^{s+1}} \leq a_7 \|u\|^a \|u\|_{L^2}^{1-a} \tag{1.50}$$

for all $u \in E$ where $2a = n(s-1)(s+1)^{-1}$. Moreover if $u \in E_{k-1}^\perp$,

$$\|u\|_{L^2} \leq \lambda_{k-1}^{-1/2} \|u\|. \tag{1.51}$$

Substituting (1.50)–(1.51) into (1.49) yields

$$J(u) \geq \tfrac{1}{2}\rho^2 - \frac{\alpha_2}{\lambda_k}\rho^2 - \alpha_3 \lambda_k^{-(1-a)(s+1)/2} \rho^{s+1} - \alpha_4 - \|f\|_{L^2} \lambda_k^{-1/2} \rho. \tag{1.52}$$

Choose $k_0$ so large that $4\alpha_2 \leq \lambda_k$ and $\rho = \rho_k$ so that

$$\rho_k = \tfrac{1}{8} \lambda_k^{((1-a)/2)((s+1)/(s-1))}. \tag{1.53}$$

Therefore

$$J(u) \geq \tfrac{1}{8}\rho_k^2 - \|f\|_{L^2} \lambda_k^{-1/2} \rho_k - \alpha_4. \tag{1.54}$$

The asymptotic distribution of the eigenvalues of (1.41) is such that for large $k$,

$$\lambda_k \geq \alpha_5 k^{2/n} \tag{1.55}$$

for $\alpha_5$ independent of $k$ [**18**]. Combining (1.54)–(1.55) then gives the lemma.

To construct a sequence of critical values of $J$, another set of comparison values first must be defined. Let

$$U_k = \left\{ u = t v_{k+1} + w \mid t \in [0, R_{k+1}],\ w \in B_{R_{k+1}} \cap E_k,\ \|u\| \leq R_{k+1} \right\}$$

and

$$\Lambda_k = \left\{ H \in C(U_k, E) \mid H|_{D_k} \in \Gamma_k \text{ and } H(u) = u \right.$$
$$\left. \text{if } \|u\| = R_{k+1} \text{ or } u \in \left(B_{R_{k+1}} \setminus B_{R_k}\right) \cap E_k \right\}.$$

Now define

$$c_k = \inf_{H \in \Lambda_k} \max_{u \in U_k} J(H(u)), \qquad k \in \mathbf{N}. \tag{1.56}$$

LEMMA 1.57. *Suppose $c_k > b_k \geq M_2$. Let $\delta \in (0, c_k - b_k)$ and*

$$\Lambda_k(\delta) = \left\{ H \in \Lambda_k \mid J(H) \leq b_k + \delta \text{ on } D_k \right\}.$$

*Let*

$$c_k(\delta) = \inf_{H \in \Lambda_k(\delta)} \max_{u \in U_k} J(H(u)), \qquad k \in \mathbf{N}. \tag{1.58}$$

*Then $c_k(\delta)$ is a critical value of $J$.*

REMARK 1.59. Since by (1.58), (1.56), and (1.43), $c_k(\delta) \geq c_k \geq b_k$ and $b_k \to \infty$ as $k \to \infty$ by Lemma 1.46, the existence of a subsequence of $c_k$'s which satisfy $c_k > b_k$

then guarantees an unbounded sequence of critical values of $J$ and hence Theorem 1.5. As will be seen shortly, (1.2) implies that such a sequence of $c_k$'s exists.

For the proof of Lemma 1.57 we require the following standard "Deformation Theorem" [**14**, **19**].

LEMMA 1.60. *Let* $J \in C^1(E, \mathbf{R})$ *and satisfy* $(PS)$ *on* $\hat{A}_M$. *Then if* $c > M$, $\bar{\varepsilon} > 0$, *and* $c$ *is not a critical value of* $J$, *there exists* $\varepsilon \in (0, \bar{\varepsilon})$ *and* $\eta \in C([0, 1] \times E, E)$ *such that*

(1) $\eta(t, u) = u$ *if* $u \notin J^{-1}(c - \bar{\varepsilon}, c + \bar{\varepsilon})$,

(2) $\eta(1, A_{c+\varepsilon}) \subset A_{c-\varepsilon}$.

PROOF OF LEMMA 1.57. Note first that by the definition of $b_k$ and $\Lambda_k$, $\Lambda_k(\delta) \neq \varnothing$. Choose $\bar{\varepsilon} = \frac{1}{2}(c_k - b_k - \delta) > 0$. If $c_k(\delta)$ is not a critical value of $J$, there exists an $\varepsilon$ and $\eta$ as in Lemma 1.60. Choose $H \in \Lambda_k(\delta)$ such that

$$\max_{U_k} J(H(u)) \leqslant c_k(\delta) + \varepsilon. \tag{1.61}$$

Consider $\eta(1, H(u)) \in C(U_k, E)$. Note that if $\|u\| = R_{k+1}$ or $u \in (B_{R_{k+1}} \setminus B_{R_k}) \cap E_k$, $J(H(u)) = J(u) \leqslant 0$ so $\eta(1, H(u)) = u$ by (1) of Lemma 1.60 (since we can assume $b_k > 0$). Therefore $\eta(1, H) \in \Lambda_k$. Moreover on $D_k$, $J(H(u)) \leqslant b_k + \delta \leqslant c_k - \bar{\varepsilon} \leqslant c_k(\delta) - \bar{\varepsilon}$ via our choice of $\delta$ and $\bar{\varepsilon}$. Therefore $\eta(1, H) = H \leqslant b_k + \delta$ on $D_k$, again by (1) of Lemma 1.60. Thus $\eta(1, H) \in \Lambda_k(\delta)$ and by (1.61) and (2) of Lemma 1.60,

$$\max_{U_k} J(\eta(1, H(u))) \leqslant c_k(\delta) - \varepsilon, \tag{1.62}$$

contrary to the definition of $c_k(\delta)$. Hence $c_k(\delta)$ is a critical value of $J$.

REMARK 1.63. Note that $c_k(\delta_1) \geqslant c_k(\delta_2)$ if $\delta_1 \leqslant \delta_2$.

Now to complete the proof of Theorem 1.5, by Remark 1.59, it suffices to show that if $s$ satisfies (1.2) and $p$ satisfies $(p_1) - (p_4)$, $c_k = b_k$ is not possible for all large $k$. Indeed we have

LEMMA 1.64. *If* $c_k = b_k$ *for all* $k \geqslant k_1$, *there is a constant* $\gamma = \gamma(k_1)$ *such that*

$$b_k \leqslant \gamma k^{\mu/\mu - 1}. \tag{1.65}$$

Thus comparing (1.65) to (1.47), we see the inequalities are incompatible if

$$\frac{\mu}{\mu - 1} < \frac{(n+2) - s(n-2)}{n(s-1)}.$$

But this is precisely condition (1.2) on $s$. Thus Theorem 1.5 is proved.

PROOF OF LEMMA 1.64. Let $k \geqslant k_1$ and $\varepsilon > 0$. Choose $H \in \Lambda_k$ such that

$$\max_{u \in U_k} J(H(u)) \leqslant b_k + \varepsilon. \tag{1.66}$$

Let $\hat{H}(u) = H(u)$ if $u \in U_k$ and $\hat{H}(u) = -H(-u)$ if $-u \in U_k$. Since $H|_{B_{R_{k+1}}} \cap E_k$ is odd and continuous, $\hat{H}$ is well defined and $\hat{H} \in \Gamma_{k+1}$. Therefore

$$b_{k+1} \leqslant \max_{D_{k+1}} J(\hat{H}(u)). \tag{1.67}$$

But $D_{k+1} = U_k \cup (-U_k)$ and by Lemma 1.18 and (1.66),

$$\max_{-U_k} J(\hat{H}(u)) = b_k + \varepsilon + \beta_1(|b_k + \varepsilon|^{1/\mu} + 1). \tag{1.68}$$

Thus (1.67)–(1.68) imply

$$b_{k+1} \leqslant b_k + \varepsilon + \beta_1\left(|b_k + \varepsilon|^{1/\mu} + 1\right). \tag{1.69}$$

Since $\varepsilon > 0$ is arbitrary,

$$b_{k+1} \leqslant b_k + \beta_1\left(|b_k|^{1/\mu} + 1\right) \tag{1.70}$$

for all $k \geqslant k_1$. An easy induction argument—see e.g. **[9–10]**—then yields (1.65).

REMARK 1.71. An analysis of Theorem 1.5 shows that by slightly modifying several of the lemmas, the following result holds:

THEOREM 1.72. *Suppose p satisfies* $(p_1)$–$(p_4)$, $f(x, \xi)$ *satisfies* $(p_1)$,

$$|f(x, \xi)| \leqslant \alpha_3 + \alpha_4 |\xi|^\sigma, \qquad 0 \leqslant \sigma < \mu - 1, \tag{$f_1$}$$

*and*

$$\frac{(n+2)-(n-2)s}{n(s-1)} > \frac{\mu}{\mu - \sigma - 1}. \tag{1.73}$$

*Then the equation*

$$\begin{cases} Lu = p(x, u) + f(x, u), & x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{cases} \tag{1.74}$$

*possesses an unbounded sequence of weak solutions.*

Theorem 1.72 generalizes a result of Dong and Li **[11]**. We will not carry out the details.

REMARK 1.75. The question of whether or not the growth restrictions on $s$ (1.2) and (1.73) are essential for these results remains open. In a very interesting recent work **[12]**, Bahri has given a partial answer. He proved for

$$\begin{cases} -\Delta u = |u|^{s-1}u + f(x), & x \in \Omega, \\ \quad u = 0, \quad x \in \partial\Omega, \end{cases} \tag{1.76}$$

for the full range of $s$: $1 < s < (n+2)(n-2)^{-1}$ that there is an open dense set of $f \in L^2(\Omega)$ for which (1.76) possesses an infinite number of distinct solutions. One knows from an identity of Pohozaev **[20]** that even if $f \equiv 0$, the result is false in general if $s \geqslant (n+2)(n-2)^{-1}$.

**2. The second order Hamiltonian system case.** A result analogous to Theorem 1.5 holds for second order Hamiltonian systems. Consider such a system.

$$\ddot{q} + V'(q) = \varphi(t) \tag{2.1}$$

where $q \in \mathbf{R}^n$ and $V$ satisfies

$(V_1)$ $V \in C^1(\mathbf{R}^n, \mathbf{R})$,

$(V_2)$ There are constants $a_1, a_2 > 0$ and $\nu > 2$ such that

$$|V(q)| \leqslant a_1 + a_2|q|^\nu \quad \text{for all } q \in \mathbf{R}^n,$$

$(V_3)$ There are constants $\mu > 2$ and $\bar{q} > 0$ such that

$$0 < \mu V(q) \leqslant q \cdot V'(q) \quad \text{for all } |q| \geqslant \bar{q}.$$

In $(V_3)$ and elsewhere $p \cdot q$ denotes the usual inner product of two elements of $\mathbf{R}^n$. As in §1, $(V_3)$ implies the existence of constants $a_3, a_4, a_5 > 0$ such that

$$\frac{1}{\mu}(q \cdot V'(q) + a_3) \geqslant V(q) + a_4 \geqslant a_5 |q|^\mu \quad \text{for all } q \in \mathbf{R}^n. \tag{2.2}$$

We assume $\varphi(t)$ is periodic in $t$. Without loss of generality we can take the period to be $2\pi$. The functional corresponding to (2.1) is

$$I(q) = \int_0^{2\pi}\left[\frac{1}{2}|\dot{q}|^2 - V(q) - \varphi \cdot q\right] dt. \tag{2.3}$$

Hypotheses $(V_1)$–$(V_2)$ imply $I \in C^1(E, \mathbf{R})$ where now $E = (W^{1,2}(S^1))^n$ with the norm in $E$ given by

$$\|q\|^2 = \int_0^{2\pi}(|\dot{q}|^2 + |q|^2)\, dt.$$

Our main result is

THEOREM 2.4. *Suppose $V$ satisfies $(V_1)$–$(V_3)$, $\varphi \in (L^2(S^1))^n$, and*

$$\nu < 4\mu - 2. \tag{2.5}$$

*Then $I(q)$ has an unbounded sequence of critical values.*

As in §1, corresponding critical points are weak solutions of (2.1) and it is easy to show they satisfy (2.1) a.e. Moreover if $\varphi \in (C(S^1))^n$, then these weak solutions in fact belong to $(C^2(S^1))^n$.

When $\varphi \equiv 0$, it is known that Theorem 2.4 is true solely under hypotheses $(V_1)$ and $(V_3)$ **[8]**. A result like Theorem 2.4 has recently been announced by Bahri and Berestycki **[13]** who further require $V \in C^2$ and in place of (2.5) have the more stringent condition $\nu < 2\mu$.

Our proof of Theorem 2.4 closely parallels that of Theorem 1.5. Therefore we will be somewhat sketchy in our exposition here.

LEMMA 2.6. *If $q$ is a critical value of $I$, there is a constant $a_6$ depending on $\|\varphi\|_{L^2}$ such that*

$$\int_0^{2\pi}(V(q) + a_4)\, dt \leqslant \frac{1}{\mu}\int_0^{2\pi}(q \cdot V'(q) + a_3)\, dt \leqslant a_6\left(I^2(q) + 1\right)^{1/2}. \tag{2.7}$$

PROOF. As in Lemma 1.8.

Now we set up a modified problem for (2.1). Let $\chi$, $\mathcal{G}$ be as in §1 (with $I(u)$ replaced by $I(q)$ given by (2.3)). Let

$$\psi(q) = \chi\left(\mathcal{G}(q)^{-1}\int_0^{2\pi}(V(q) + a_4)\, dt\right)$$

and set

$$J(q) = \int_0^{2\pi}\left[\frac{1}{2}|\dot{q}|^2 - V(q) - \psi(q)\varphi \cdot q\right] dt. \tag{2.8}$$

LEMMA 2.9. *If $q \in \operatorname{supp} \psi$, then*

$$\left|\int_0^{2\pi}\varphi \cdot q\, dt\right| \leqslant \alpha_1\left(|I(q)|^{1/\mu} + 1\right) \tag{2.10}$$

*where $\alpha_1$ depends on $\|\varphi\|_{L^2}$.*

PROOF. As in Lemma 1.13.

For $\theta \in [0, 2\pi)$, let $(T_\theta q)(t) = q(t + \theta)$.

LEMMA 2.11. *There is a constant $\beta_1$ depending on $\|\varphi\|_{L^2}$ such that for all $q \in E$ and $\theta \in [0, 2\pi)$,*

$$|J(q) - J(T_\theta q)| \leqslant \beta_1\big(|J(q)|^{1/\mu} + 1\big). \tag{2.12}$$

PROOF. Observing that $\|q\|_{L^2} = \|T_\theta q\|_{L^2}$, the proof is essentially as in Lemma 1.18.

LEMMA 2.13. *There are constants $M_0, \alpha_0 > 0$ and depending on $\|\varphi\|_{L^2}$ such that whenever $M \geqslant M_0$, $J(q) \geqslant M$, and $q \in \operatorname{supp} \psi$, then $J(q) \geqslant \alpha_0 M$.*

PROOF. As in Lemma 1.25.

LEMMA 2.14. *There is a constant $M_1 > 0$ such that $J(q) \geqslant M_1$ and $J'(q) = 0$ implies $J(q) = I(q)$ and $I'(q) = 0$.*

PROOF. As in Lemma 1.29.

Let $\hat{A}_c = \{q \in E \mid J(q) \geqslant c\}$ and $A_c = \{q \in E \mid J(q) \leqslant c\}$.

LEMMA 2.15. *$J \in C^1(E, \mathbf{R})$ and there exists a constant $M_2 > 0$ such that $J$ satisfies $(PS)$ on $\hat{A}_{M_2}$.*

PROOF. $J \in C^1(E, \mathbf{R})$ follows from $(V_1)$, $(V_2)$ and the smoothness and form of $\psi$. To verify (PS), we argue in a similar fashion to Lemma 1.34. As in (1.36) with $\rho$ chosen to satisfy (1.37), we get

$$\begin{aligned} \rho\|q_m\| + K &\geqslant \varepsilon\int_0^{2\pi}|\dot{q}_m|^2\,dt + \frac{\mu\varepsilon}{2}a_5\|q_m\|_{L^\mu}^{\mu} - a_7\|q_m\|_{L^2} \\ &= \varepsilon\|q_m\|^2 + \frac{\mu\varepsilon a_5}{2}\|q_m\|_{L^\mu}^{\mu} - \varepsilon\|q_m\|_{L^2}^2 - a_7\|q_m\|_{L^2} - a_8. \end{aligned} \tag{2.16}$$

Hence using the Hölder and Young inequalities as in (1.38), we conclude $\{q_m\}$ is uniformly bounded in $E$. Writing

$$\begin{aligned} J'(q_m)Q = {}&\big(1 + T_1(q_m)\big)\int_0^{2\pi}\big(\dot{q}_m \cdot \dot{Q} + q_m \cdot Q\big)\,dt \\ &- \big(1 + T_1(q_m)\big)\int_0^{2\pi} q_m \cdot Q\,dt + \text{lower order terms} \end{aligned} \tag{2.17}$$

we see

$$J'(q_m) = \big(1 + T_1(q_m)\big)q_m + \mathcal{P}(q_m)$$

where $\mathcal{P}$ is compact. Thus the argument of Lemma 1.34 shows $(q_m)$ has a convergent subsequence and (PS) is satisfied.

As a consequence of Lemma 2.14, in order to prove Theorem 2.4, it suffices to show $J$ has an unbounded sequence of critical values. This will be accomplished as in §1 by a comparison argument. Let $e_1, \ldots, e_n$ denote the usual orthonormal basis in $\mathbf{R}^n$. Define $v_{jk} = (\sin jt)e_k$ and $w_{jk} = (\cos jt)e_k$ for $j \in \mathbf{N} \cup \{0\}$ and $1 \leqslant k \leqslant n$. These functions form an orthogonal basis for $E$. Let

$$E_{mi} = \operatorname{span}\{v_{jk}, w_{jk} \mid 0 \leqslant j \leqslant m,\ 1 \leqslant k \leqslant i\}$$

where $1 \leqslant i \leqslant n$. By (2.2) there exists $R_{mi} > 0$ such that $J|_{E_{mi}} \leqslant 0$ if $\|q\| \geqslant R_{mi}$. Let $D_{mi} = B_{R_{mi}} \cap E_{mi}$. We say $h \in C(D_{mi}, E)$ is *equivariant* if $h(T_\theta q) = T_\theta h(q)$ for all $\theta \in [0, 2\pi]$. Let

$$\Gamma_{ki} = \{h \in C(D_{ki}, E) \mid h \text{ is equivariant and } h(q) = q \text{ whenever } \|q\| = R_{ki} \text{ or } q \in E_{0n}\}. \tag{2.18}$$

Define

$$b_{ki} = \inf_{h \in \Gamma_{ki}} \max_{q \in D_{ki}} J(h(q)). \tag{2.19}$$

Let $E_{k,i-1}^{\perp}$ denote the orthogonal complement of $E_{k,i-1}$ if $i \neq 1$ and $E_{k,0}^{\perp} \equiv E_{k-1,n}^{\perp}$.

LEMMA 2.20. *For all $k \in \mathbf{N}$, $1 \leqslant i \leqslant n$, $\rho < R_{ki}$, and $h \in \Gamma_{ki}$,*

$$h(D_{ki}) \cap \partial B_\rho \cap E_{k,i-1}^{\perp} \neq \varnothing. \tag{2.21}$$

PROOF. The proof of this lemma will be carried out in the Appendix.

LEMMA 2.22. *There are constants $\beta_2 > 0$ and $k_0 \in \mathbf{N}$ such that for $k \geqslant k_0$ and $1 \leqslant i \leqslant n$,*

$$b_{ki} \geqslant \beta_2 k^{(\nu+2)/(\nu-2)}. \tag{2.23}$$

PROOF. If $k \geqslant 1$ and $q \in \partial B_\rho \cap E_{k,i-1}^{\perp}$,

$$\|q\|_{L^2} \leqslant \|\dot{q}\|_{L^2}, \tag{2.24}$$

and therefore

$$\|q\| \leqslant 2\|\dot{q}\|_{L^2}. \tag{2.25}$$

Arguing as in (1.48)–(1.49) using $(V_2)$ and (2.25) leads to

$$J(q) \geqslant \tfrac{1}{4}\rho^2 - a_2\|q\|_{L^\nu}^{\nu} - \alpha_2 - \alpha_3\|q\|_{L^2}. \tag{2.26}$$

The analogues here of (1.50)–(1.51) are

$$\|q\|_{L^\nu} \leqslant a_7\|q\|^{(\nu-2)/2\nu}\|q\|_{L^2}^{(\nu+2)/2\nu} \tag{2.27}$$

for all $q \in E$ and

$$\|q\|_{L^2} \leqslant k^{-1}\|\dot{q}\|_{L^2} \tag{2.28}$$

for $q \in E_{k,i-1}^{\perp}$ (unless $i = 1$ in which case $k$ is replaced by $k - 1$). Continuing as in Lemma 1.46 then yields (2.23).

Next to construct critical values of $J$, let

$$U_{ki} = \{q = \tau v_{k,i+1} + Q \mid \tau \in [0, R_{k,i+1}],\ Q \in B_{R_{k,i+1}} \cap E_{k,i},\ \|q\| \leqslant R_{k,i+1}\}$$

where if $i = n$, $v_{k,i+1} \equiv v_{k+1,1}$ and $R_{k,i+1} \equiv R_{k+1,1}$. Let

$$\Lambda_{ki} = \{H \in C(U_{ki}, E) \mid H|_{D_{ki}} \in \Gamma_{ki} \text{ and } H(q) = q \text{ when } \|q\| = R_{k,i+1}, \text{ or } q \in (B_{R_{k,i+1}} \setminus B_{R_{k,i}}) \cap E_{ki}\}$$

and define

$$c_{ki} = \inf_{H \in \Lambda_{ki}} \max_{q \in U_{ki}} J(H(q)).$$

LEMMA 2.29. *Suppose* $c_{ki} > b_{ki} \geqslant M_2$. *Let* $\delta \in (0, c_{ki} - b_{ki})$ *and*

$$\Lambda_{ki}(\delta) = \{H \in \Lambda_{ki} \mid J(H) \leqslant b_{ki} + \delta \text{ on } D_{ki}\}$$

*Set*

$$c_{ki}(\delta) = \inf_{H \in \Lambda_{ki}(\delta)} \max_{q \in U_{ki}} J(H(q)). \tag{2.30}$$

*Then* $c_{ki}(\delta)$ *is a critical value of* $J$.

PROOF. Essentially as in Lemma 1.57.

LEMMA 2.31. *If* $c_{ki} = b_{ki}$ *for all* $k \geqslant k_1$ *and* $1 \leqslant i \leqslant n$, *then there exists* $\gamma = \gamma(k_1)$ *such that*

$$b_{ki} \leqslant \gamma k^{\mu/\mu - 1}. \tag{2.32}$$

PROOF. Let $k \geqslant k_1$, $1 \leqslant i \leqslant n$, $\varepsilon > 0$, and $H \in \Lambda_{ki}$ such that

$$\max_{U_{ki}} J(H(q)) \leqslant b_{ki} + \varepsilon. \tag{2.33}$$

Let $\hat{H}(q) = H(q)$ for $q \in U_{ki}$ and $\hat{H}(T_\theta q) = T_\theta H(q)$ for $q \in U_{ki}$. Note that $\{T_\theta U_{ki} \mid \theta \in [0, 2\pi]\} = D_{k,i+1}$ and by construction $\hat{H}$ is equivariant. Moreover since $H \in C(D_{ki}, E)$, $\hat{H} \in C(D_{k,i+1}, E)$ and $\hat{H}(q) = q$ if $\|q\| = R_{k,i+1}$. Therefore $\hat{H} \in \Gamma_{k,i+1}$. Now arguing as in Lemma 1.64 with Lemma 1.18 replaced by Lemma 2.11, we find

$$b_{k,i+1} \leqslant b_{ki} + \beta_1\big(|b_{ki}|^{1/\mu} + 1\big) \tag{2.34}$$

where $b_{k,i+1} = b_{k+1,1}$ if $i = n$. A slight extension of the argument of [**9** or **10**] then yields (2.32).

The proof of Theorem 2.4 is now immediate on comparing (2.32) to (2.13) and recalling (2.5).

REMARK 2.35. As in §1, a more general perturbation than $\varphi$ may be permitted using the above arguments. Indeed we have

THEOREM 2.36. *If* $V$ *satisfies* $(V_1)$–$(V_3)$, $\varphi(t, q) \in C([0, 2\pi] \times \mathbf{R}^n, \mathbf{R}^n)$ *is* $2\pi$ *periodic in* $t$ *and*

$$|\varphi(t, q)| \leqslant \alpha_3 + \alpha_4 |q|^\sigma$$

*where* $0 \leqslant \sigma < \mu - 1$ *and* $\nu < 4\mu(\sigma + 1)^{-1} - 2$, *then the system*

$$\ddot{q} + V'(q) = \varphi(t, q)$$

*has an unbounded sequence of (classical) solutions.*

We omit the details.

APPENDIX. Our goal here is to prove

LEMMA 2.20. *For all* $k \in \mathbf{N}$, $k \geqslant 1$, $1 \leqslant i \leqslant n$, $\rho < R_{ki}$, *and* $h \in \Gamma_{ki}$,

$$h(D_{ki}) \cap \partial B_\rho \cap E^{\perp}_{k,i-1} \neq \varnothing. \tag{2.21}$$

The analogous result in §1, Lemma 1.44, was proved with the aid of the Borsuk-Ulam Theorem. The proof of Lemma 2.20 in turn depends on an $S^1$ version of the Borsuk-Ulam Theorem. In [**21**], the following situation was studied: Let $S^1$ act on $\mathbf{R}^l \times \mathbf{R}^{2k}$ via a family of orthogonal transformations such that Fix $S^1 = \mathbf{R}^l \times \{0\}$. For $j < k$ we consider $\mathbf{R}^{2j}$ to be a subspace of $\mathbf{R}^{2k}$ via $\mathbf{R}^{2k} = \mathbf{R}^{2j} \times \mathbf{R}^{2(k-j)} \supset \mathbf{R}^{2j} \times \{0\}$. It was shown in **[21]** that

LEMMA A-1. *Let $\Omega$ be a bounded invariant neighborhood of* 0 *in* $\mathbf{R}^l \times \mathbf{R}^{2k}$ *and let* $f \in C(\partial\Omega, R^l \times \mathbf{R}^{2j})$ *where* $j < k$ *and* $f$ *is equivariant. Suppose further that* $f|_{(S^{l-1}\times\{0\})\cap\partial\Omega}$ *is the identity. Then* $\{x \in \partial\Omega \mid f(x) = 0\}$ *is nonempty.*

REMARK. If $l = 0$ and Fix $S^1 = \{0\}$, it is easy to use e.g. the index theory of **[15]** to prove Lemma A-1.

We will show how to use Lemma A-1 to prove Lemma 2.20. First the special case.

LEMMA A-2. *Let* $\mathfrak{M} = \{\Phi \in \Gamma_{ki} \mid \Phi(D_{ki}) \subset E_{mj}$ *for some* $m$ *and* $j\}$. *Then* (2.21) *holds for all* $\Phi \in \mathfrak{M}$.

PROOF. Let $h \in \mathfrak{M}$. Then $h^{-1}(B_\rho)$ is a neighborhood of 0 in $E_{ki}$. Let $\Omega$ be the component of $h^{-1}(\partial B_\rho)$ containing 0. Then $\Omega \subset D_{ki}$ is an invariant neighborhood of 0 in $E_{ki}$ (i.e. $x \in \Omega$ implies $T_\theta x \in \Omega$ $\forall\theta \in [0, 2\pi)$). Let $P_{k,i-1}$, $P^\perp_{k,i-1}$ denote respectively the orthogonal projection of $E$ onto $E_{k,i-1}$, $E^\perp_{k,i-1}$ respectively. Then $P_{k,i-1}h \equiv f \in C(\partial\Omega, E_{k,i-1})$. Since $E_{k,i-1}$ is an invariant subspace of $E$, $f$ is an equivariant map. Note that

$$E_{0n} = \{q \in E \mid T_\theta q = q \text{ for all } \theta \in [0, 2\pi)\} = \text{Fix } S^1.$$

Since $h \in \mathfrak{M}$, $h(q) = q = f(q)$ on $E_{0n} \cap D_{ki}$. With some obvious identifications we have satisfied the hypotheses of Lemma A-1. Hence $f$ has a zero $Q$ on $\partial\Omega$. Consequently $h(Q) = P^\perp_{k,i-1}h(Q) \in \partial B_\rho \cap E^\perp_{k,i-1}$. Thus (2.21) is satisfied.

Now we can give the

PROOF OF LEMMA 2.20. Let $h \in \Gamma_{ki}$ and $m > k$. Then $P_{mi}h \in \mathfrak{M}$. By Lemma A-1,

$$P_{mi}h(D_{ki}) \cap \partial B_\rho \cap E^\perp_{k,i-1} \neq \varnothing.$$

Therefore there is a sequence of $m$'s $\to \infty$ and corresponding to $q_m \in D_{ki}$ such that

$$P_{mi}h(q_m) \in \partial B_\rho \cap E^\perp_{k,i-1}. \tag{A-2}$$

Passing to a subsequence if necessary, the compactness of $D_{ki}$ implies $q_m \to q \in D_{ki}$. Since

$$\|h(q) - P_{mi}h(q_m)\| \leq \|h(q) - P_{mi}h(q)\| + \|P_{mi}(h(q) - h(q_m))\| \to 0$$

as $m \to \infty$, by (A-2)

$$h(q) \in \partial B_\rho \cap E^\perp_{k,i-1}$$

and (2.21) is satisfied.

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